King Fahd University of Petroleum & Minerals
College of Computer Sciences & Engineering
Department of Information and Computer Science

ICS 253: Discrete Structures I
Final Exam – 141

120 Minutes

Calculators, mobile phones and other electronic devices are not permitted.

Question	Max	Earned	CLO*	Question	Max	Earned	CLO*
1	5		1	12	5		3
2	5		1	13	5		3
3	5		1	14	5		3
4	4		2	15	5		3
5	4		2	16	4		3
6	4		2	17	4		3
7	5		2	18	4		3
8	5		3	19	4		3
9	5		3	20	6		3
10	5		3	21	6		3
11	5		3	**Total**	**100**		**1, 2, 3**

* **CLO** Course Learning Outcomes

Monday, January 5, 2015

Sample Solution

Question 1: [5 Points] Logic and Proofs

Select the Boolean expression that is equal to the expression [~(~p∧q) ∧ ~(~p∧~q)] ∨ (p∧r) form the following. Circle the correct answer.

(a) p ∧ r (b) p ∨ q (c) r **(d)** p

Question 2: [5 Points] Logic and Proofs

Circle the correct answer. Select one statement of the following statements that is equivalent to the statement

"If *n* is divisible by 30, then *n* is divisible by 2, by 3 and by 5."

(a) If *n* is not divisible by 30 then *n* is divisible by 2 or divisible by 3 or divisible by 5.

(b) If *n* is not divisible by 30 then *n* is not divisible by 2 or not divisible by 3 or not divisible by 5.

(c) If *n* is divisible by 2 and divisible by 3 and divisible by 5 then *n* is divisible by 30.

(d) If *n* is not divisible by 2 or not divisible by 3 or not divisible by 5 then *n* is not divisible by 30.

Question 3: [5 Points] Logic and Proofs

What is the contrapositive statement of the statement,

"You win the game if you know the rules, and you are not overconfident."

Solution: If you lose the game then you don't know the rules or you are overconfident.

Question 4: [4 Points] Sets and Sets Operations

Circle the most proper answer. The power set P((A × B) ∪ (B × A)) has the same number of elements as the power set P((A × B) ∪ (A × B)) if and only if

(a) A = B (b) A = ∅ or B = ∅

(c) A = ∅ or B = ∅ or A = B (d) B = ∅ or A = B

Question 5: [4 Points] Functions

Define $f(n) = \frac{n}{2} + \frac{1-(-1)^n}{4}$ for all $n \in \mathbb{Z}$. Thus, $\mathbb{Z} \rightarrow \mathbb{Z}, \mathbb{Z}$ the set of all integer. Determine if f is a function. If it is not a function show why. If it is a function determine if it is onto and/or one-to-one.

Solution: f is a function and is onto but not one-to-one.

Question 6: [4 Points] Sequences and Summations

Describe the sequences 0, 0, 0, 1, 1, 1, 2, 2, 2, 3, 3, 3 by an explicit formula.

Solution: $g_n = \left\lfloor \frac{n}{3} \right\rfloor, (n \geq 0)$

Question 7: [5 Points] Induction and Recursion

We are going to prove by induction that for all integers $k \geq 1$, $\sqrt{k} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}}$. Clearly this is true for $k = 1$, as $\sqrt{1} \leq \frac{1}{\sqrt{1}}$. Assume the Induction Hypothesis (IH) that

$$\sqrt{n} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$$

Complete the remaining parts of the proof.

Solution:

We need to show that $\sqrt{n+1} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}}$ is also true.

From IH $\sqrt{n} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$ we can add for both sides $\frac{1}{\sqrt{n+1}}$

$$\sqrt{n} + \frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}}$$

$$\sqrt{n} + \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n}\sqrt{n+1} + 1}{\sqrt{n+1}} = \frac{\sqrt{n^2+n} + 1}{\sqrt{n+1}} < \frac{\sqrt{n^2} + 1}{\sqrt{n+1}} = \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}$$

This completes the induction step and hence $\sqrt{n} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$ is true for all $n \geq 1$

Question 8: [5 Points] Counting and the Pigeonhole Principle

There are K people in a room, each person picks a day of the year to get a free dinner at a fancy restaurant. Find the smallest K such that there must be at least one group of six people who select the same day. Assume the year has 366 days (leap year).

Solution:

The worst case is each person selects different day. To have 6 persons who select the same day, you need to assume the worst case for the first 5 and add 1 to it.

So the answer is 366 * 5 + 1 = 1831

Question 9: [5 Points] Permutations and Combinations

How many strings of four decimal digits end with an even digit? (Leading 0 is allowed, e.g. 0892)

Solution:

We have 10 choices each for the first three digits, then 5 for the final digit, for $(10^3)(5) = 5000$ strings.

Question 10: [5 Points] Permutations and Combinations

The owner of a pizzeria prepares every pizza by always combining 4 different ingredients. How many ingredients does he need, at least, if he would like to offer 30 different pizzas in the menu?

Solution:

Using 6 different ingredients, he can obtain $\binom{6}{4} = 15$ different pizzas, while with 7 ingredients he gets $\binom{7}{4} = 35$ different pizzas. Then, since $\binom{6}{4} < 30 < \binom{7}{4}$, he needs at least 7 ingredients.

Question 11: [5 Points] Probability

The random variable X on a sample space S = {1, 2, 4, 10} has the following distribution:

X	1	2	4	10
P(x)	0.3	0.2	0.2	?

What is P(X = 10)?

Solution: 0.3

Question 12: [5 Points] Probability

A professor randomly selects three new teaching assistants from a total of 10 applicants: 6 male and 4 female students. What is the probability that no females are hired?

Solution:

$$\frac{\binom{6}{3}}{\binom{10}{3}} = \frac{1}{6} = 0.167$$

Question 13: [5 Points] Probability

A car pool contains 8 Fords (4 red and 4 white) and 12 Pontiacs (2 red and 10 white). You are allocated a car at random. You see from a distance that it is red. What is the probability that you have been given a Ford?

Solution:

Let F and R be the event of being a Ford and a red car, respectively. We are asked for P(F|R). There are 20 cars of which 8 are Fords, so P(F) = 8/20 = 0.4, and 6 cars are red, so P(R) = 6/20 = 0.3. The probability of a red car, given that it is a Ford, is P(R|F) = 4/8 = 0.5.

So

$$\mathrm{P(F|R)} = \frac{P(F).P(R|F)}{P(R)} = \frac{(0.4)(0.5)}{0.3} = 0.667$$

Question 14: [5 Points] Probability

What is the conditional probability that a family with two children has two boys, given they have at least one boy? Assume that each of the possibilities *BB*, *BG*, *GB*, and *GG* is equally likely, where *B* represents a boy and *G* represents a girl. (Note that *BG* represents a family with an older boy and a younger girl while *GB* represents a family with an older girl and a younger boy.)

Solution:

Let *E* be the event that a family with two children has two boys, and let *F* be the event that a family with two children has at least one boy. It follows that $E = \{BB\}$, $F = \{BB, BG, GB\}$, and $E \cap F = \{BB\}$. Because the four possibilities are equally likely, it follows that $p(F) = 3/4$ and $p(E \cap F) = 1/4$.We conclude that

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Question 15: [5 Points] Probability

Suppose that 100 people enter a contest and that different winners are selected at random for first, second, and third prizes. What is the probability that Ali, Salem, and Ahmad each win a prize if each has entered the contest? Show your answer using combination and division of numbers.

Solution:

Any group of 3 people is equally likely to win the 3 prizes. There are $\binom{100}{3}$ groups of 3 people, so the probability is $\frac{1}{\binom{100}{3}}$

Question 16: [4 Points] Linear Recurrence Relations

A piece of paper is 1 unit thick. By folding into half, the thickness becomes 2 units. Folding into half again, its thickness becomes 4 units, and so on. What is the thickness of the paper after it is folded 10 times?

Solution:

When it is folded 3 times, the thickness becomes 8 units. Similarly, when folded 4, 5, 6 times, the thickness becomes 16, 32 and 64 units respectively. You notice that each time the paper is folded, its thickness doubles, so you just multiply the thickness by 2 each time. Continuing in the same way, it is easy to get the answer, namely, when folded 10 times, the thickness will become 2^{10}=1024 units.

Question 17: [4 Points] Binomial Coefficients and Identities

What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x - 3y)^{200}$? Show your answer using combination and multiplication of numbers (no need to calculate a final answer).

Solution:

Using the binomial formula taking a=2x, b=-3y, n=200 and k=99 we have that $x^{101}y^{99}$ term is $\binom{200}{99}(2x)^{101}(-3y)^{99} = -\binom{200}{99}2^{101}3^{99}x^{101}y^{99}$

Hence the desired coefficient is $-\binom{200}{99}2^{101}3^{99}$

Question 18: [4 Points] Binomial Coefficients and Identities

The row of Pascal's triangle containing the binomial coefficients $\binom{6}{k}$, $0 \le k \le 6$, is:

1 6 15 20 15 6 1

Use Pascal's identity to produce the row immediately following this row in Pascal's triangle.

Solution:

By Pascal's identity, the first half of next row is

1 1+6 6+15 15+20 20+15 · · ·

With the rest determined by symmetry. Thus, the next row is

1 7 21 35 35 21 7 1

Question 19: [4 Points] Linear Recurrence Relations

Which of the following is a linear homogeneous recurrence relation with constant coefficients? Circle the most proper answer.

(a) $a_n = a_{n-1} + n$

(b) $a_n = a_{n-5} + 5$

(c) $a_n = a_{n-1} - 4a_{n-2} + a_{n-3}^3$

(d) $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$

Question 20: [6 Points] Linear Recurrence Relations

a) **[4 Points]** Find a recurrence relation for the number of ways to climb *n* stairs if the person climbing the stairs can take one stair or two stairs at a time, and state the initial conditions.

Solution:

$a_n = a_{n-1} + a_{n-2}$

$a_1 = 1, a_2 = 2$

b) **[2 Points]** In how many ways can this person climb a flight of five stairs?

$a_3 = 2 + 1 = 3, a_4 = 3 + 2 = 5, a_5 = 5 + 3 = 8$

Question 21: [6 Points] Linear Recurrence Relations

Solve the recurrence relation $a_n = 3a_{n-1} + 4a_{n-2}$ for $n \ge 2$ given the initial conditions $a_0 = 6$ and $a_1 = 8$.

Solution:

$c_1 = -3, c_2 = -4,$

$r^2 - 3r - 4 = 0$

=> $r_1 = 4$ $r_2 = -1$

=> $a_n = b_1 * 4^n + b_2 * (-1)^n$

with the initial condition: $a_0 = 6, a_1 = 8$

=> $6 = b_1 * 4^0 + b_2 * (-1)^0 = b_1 + b_2$

$8 = b_1 *4^1 + b_2 (-1)^1 = 4b_1 - b_2$

=> $b_1 = 16/5, b_2 = 14/5$

=> $a_n = 14/5 * 4^n + 16/5 * (-1)^n$